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Stability of a potential vortex with a non-rotating and rigid-body rotating top-hat jet core

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The stability of a potential vortex with a rotating and a non-rotating jet core is analysed. Eigenvalues are calculated numerically for different values of the ratio of the strength of the vortex to the axial velocity. These results show that the potential vortex becomes unstable in the presence of a jet.

1. Introduction

Since the introduction of large aircraft for commercial transport, the stability of vortex flows has become of some importance. In particular, trailing vortices from the wing tips of large aeroplanes pose a threat to light planes and hence it is of interest to study the decay of such flows.

The first investigation of circular vortex flows was done by Kelvin (1880), who obtained approximate expressions for the simple harmonic frequencies.

Rayleigh (1916) derived a necessary and sufficient condition for the stability of inviscid revolving fluids, namely that the circulation must increase outwards. This criterion applied to an infinite potential vortex predicts stability.

Ponstein (1959) examined the instability of rotating cylindrical jets including the effects of viscosity, surface tension and the surrounding air. He showed that, if the viscosity and the influence of the surrounding air are neglected, non-rotationally symmetric disturbances are sometimes more unstable than rotationally symmetric ones, and that the rotating jet becomes more stable as it rotates faster.

The stability of axisymmetric jets was investigated by Batchelor & Gill (1962), who derived a necessary condition for the amplification of the disturbances. The condition shows that a top-hat jet profile is unstable for both axisymmetric and non-axisymmetric disturbances.

Many other investigators have studied the stability of vortex flows. Crow (1970) analysed the stability of a pair of trailing vortices during the early growth stage. Both symmetric and antisymmetric eigenmodes were shown to be unstable. Parks (1970) modified Crow's theory to take into account finite core radii. Widnall & Bliss (1971) considered this same problem for a long wavelength disturbance only.

In this paper, we consider the stability of a potential vortex in the presence of a rotating and non-rotating axial jet. The influence of the ratio of the strength of the vortex to the axial velocity of the jet has been determined. Numerical values of the growth rates have been obtained, and it is shown that a rotating axial jet makes the potential vortex unstable.

2. Mathematical analysis

The equations governing the behaviour of an inviscid incompressible fluid read

$$\nabla . \mathbf{U} = 0, \tag{1}$$

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla(\tfrac{1}{2}\mathbf{U}.\mathbf{U}) - \mathbf{U}\times\nabla\times\mathbf{U} = -\frac{1}{\rho}\nabla P. \tag{2}$$

As only non-radial mean flows are considered, the velocity components in the r, θ and z directions, respectively, may be written as

$$U_1 = u', \quad U_2 = V+v', \quad U_3 = W+w', \tag{3}$$

with $\quad V = V(r), \quad W = \text{constant}, \quad u',v',w' = u',v',w'(r,\theta,z,t),$

where V and W are mean flow velocities and the 'primed' quantities represent disturbance velocities.

If we assume the disturbances to be of the following form:

$$\{p'/\bar{\rho}, u', v', w'\} = \{\pi(r), f(r), g(r), h(r)\}\exp[in\theta + i\alpha(z-ct)], \tag{4}$$

where p' is the pressure disturbance and $\bar{\rho}$ the density of the fluid, the linearized disturbance equations are

$$\frac{1}{r}\frac{d}{dr}(rf) + in\frac{g}{r} + i\alpha h = 0, \quad \text{(continuity)}, \tag{5}$$

$$\left.\begin{aligned} -\frac{d}{dr}\pi &= \left\{if\left(\frac{V}{r}n + (W-c)\alpha\right) - 2g\frac{V}{r}\right\} && (6)\\ -in\frac{\pi}{r} &= \left\{ig\left(\frac{V}{r}n + (W-c)\alpha\right) + f\left(\frac{dV}{dr} + \frac{V}{r}\right)\right\} && (7)\\ -i\alpha\pi &= \{ih[(V/r)n + (W-c)\alpha]\} && (8)\end{aligned}\right\} \text{(momentum)}.$$

The mean velocity components for the jet and the potential vortex are assumed to be of the form

$$V = \begin{cases}\Omega r, \\ \Gamma/r,\end{cases} \quad W = \begin{cases}\text{constant} & \text{for } 0 \leqslant r \leqslant r_0, & (9)\\ 0 & \text{for } r \geqslant r_0. & (10)\end{cases}$$

For the region $0 \leqslant r \leqslant r_0$, the linearized equations (5)–(8) can be combined to give

$$\frac{d}{dr}\left(r\frac{dh}{dr}\right) + \left(\mu^2 - \frac{n^2}{r}\right)h = 0, \tag{11}$$

with $$\mu^2 = \alpha^2\{1 - 4\Omega^2/[\Omega n + (W-c)\alpha]^2\}. \tag{12}$$

The only solution bounded at $r = 0$ is a modified Bessel function of the first kind:

$$h = C_1 I_n(\mu r), \tag{13}$$

and therefore
$$\pi = -\alpha^{-1}[\Omega n + (W-c)\alpha]\,C_1 I_n(\mu r), \tag{14}$$
$$f = -\frac{i\alpha C_1}{\mu^2(\Omega n + (W-c)\alpha)}\left\{\frac{2\Omega n}{r} I_n(\mu r) + (\Omega n + (W-c)\alpha)\frac{d}{dr} I_n(\mu r)\right\}, \tag{15}$$
where C_1 is an arbitrary constant.

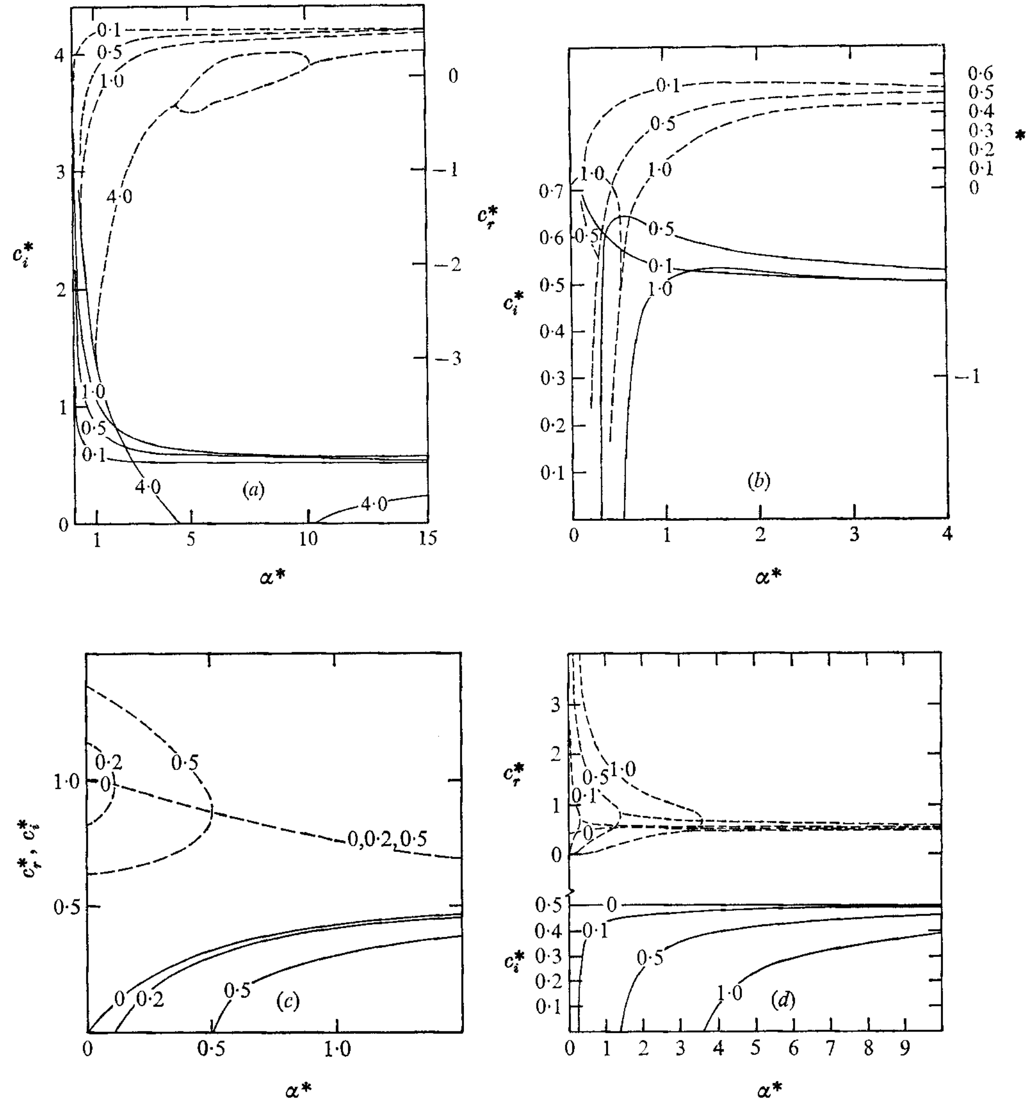
In the case of no rigid-body rotation ($\Omega = 0$), $\mu = \alpha$ and one obtains
$$h = C_1 I_n(\alpha r), \tag{16}$$
$$f = -iC_1 \frac{1}{\alpha}\frac{d}{dr} I_n(\alpha r), \tag{17}$$
$$\pi = -(W-c)\,C_1 I_n(\alpha r). \tag{18}$$
For the outer region $r \geqslant r_0$ the equation governing the axial velocity disturbance h is
$$\frac{d}{dr}\left(r\frac{dh}{dr}\right) - \left(\alpha^2 r + \frac{n^2}{r}\right) h = 0. \tag{19}$$

The only solution bounded at infinity is a Bessel function of second kind:
$$h(r) = C_2 K_n(\alpha r) \tag{20}$$
and
$$\pi = \frac{1}{\alpha}\left(\alpha c - \frac{\Gamma n}{r^2}\right) C_2 K_n(\alpha r), \tag{21}$$
$$f = -\frac{i}{\alpha}\frac{d}{dr} K_n(\alpha r). \tag{22}$$

We consider the system formed by a rotating jet for $r < r_0$ and a potential vortex for $r \geqslant r_0$. The flow being inviscid, the discontinuity at $r = r_0$ is regarded as a cylindrical vortex sheet of infinite vorticity.

The radius of the perturbed vortex sheet can be expressed as
$$R = r_0 + \delta, \quad \text{where} \quad \delta = A \exp i[n\theta + \alpha(z - ct)].$$
The steady-state pressures can be obtained from (2). They are
$$\bar{P}_1 = \tfrac{1}{2}\bar{\rho}\Omega^2 r^2 + C_1' \quad \text{for the jet}$$
and
$$\bar{P}_2 = -\tfrac{1}{2}\bar{\rho}\Gamma^2/r^2 + C_2' \quad \text{for the vortex.}$$
In the unperturbed state, the two pressures are equal on the surface of discontinuity. In the perturbed state, the pressure must be continuous through the vortex sheet:
$$\bar{P}_1(r_0+\delta) + p_1' = \bar{P}_2(r_0+\delta) + p_2'. \tag{23}$$
Linearizing and substituting for the disturbances we get
$$\Omega^2 r_0 A - \alpha^{-1}[\Omega n + (W-c)\alpha]\,C_1 I_n(\mu r_0) - \left(\frac{\Gamma^2}{r_0^3} A + \frac{1}{\alpha}\alpha c - \frac{\Gamma n}{r_0^2} C_2 K_n(\alpha r_0)\right) = 0. \tag{24}$$

The rate of change of displacement of the vortex sheet must equal the radial velocity in the vortex and the jet:
$$\frac{D\delta}{Dt} = \frac{\partial\delta}{\partial t} + (\mathbf{U}.\nabla)\,\delta = u'. \tag{25}$$



FIGURES 1 (*a–d*). For legend see facing page.

From (9), (15), (21) and (25), it follows that

$$i\alpha A\left[\frac{\Gamma n}{\alpha r_0^2}-c\right]+\frac{i}{\alpha}C_2\frac{d}{dr}K_n(\alpha r)\bigg|_{r=r_0}=0, \tag{26}$$

$$iA[\Omega n+(W-c)\alpha]=\frac{-i\alpha C_1}{\mu^2(\Omega n+(W-c)\alpha)}\left[\frac{n}{r_0}2\Omega I_n(\mu r_0)+(\Omega n+(W-c)\alpha)\frac{d}{dr}I_n(\mu r)\bigg|_{r=r_0}\right]. \tag{27}$$

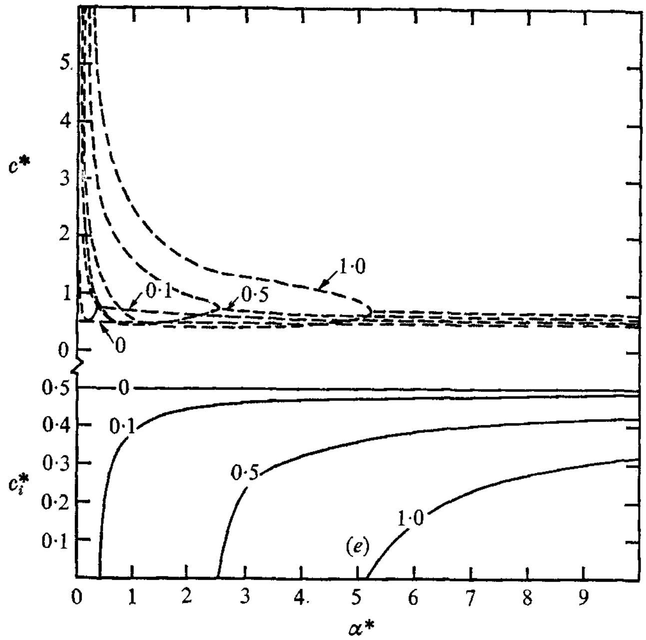


FIGURE 1. Phase velocity and growth rate for (*a*) $n = -2$, (*b*) $n = -1$, (*c*) $n = 0$, (*d*) $n = 1$ and (*e*) $n = 2$ for different values of Γ^* (numbers on curves) in the non-rotating case ($\Omega^* = 0$). ---, c_r^*; ——, c_i^*.

Equations (24), (26) and (27) give the following eigenvalue equation:

$$\frac{\alpha^*}{\mu^{*2}}\left[\frac{2n\Omega^*}{\alpha^*(1-c^*)+\Omega^* n}+\mu^*\frac{I_n'(\mu^*)}{I_n(\mu^*)}\right]\left[(\Gamma^* n-\alpha^* c^*)^2\frac{1}{\alpha^*}\frac{K_n(\alpha^*)}{K_n'(\alpha^*)}-\Omega^{*2}+\Gamma^{*2}\right]$$
$$-\alpha^{*-1}[\Omega^* n+(1-c^*)\alpha^*]^2=0, \quad (28)$$

where $\alpha^* = \alpha r_0$, $c^* = c/W$, $\Gamma^* = \Gamma/r_0 W$, $\Omega^* = \Omega r_0/W$,

$$\mu^{*2} = \alpha^{*2}\left[1-4\Omega^{*2}/(\Omega^* n+(1-c^*)\alpha^*)^2\right]$$

(a prime denotes differentiation with respect to the argument). This equation reduces to that obtained by Batchelor & Gill for $\Omega^* = \Gamma^* = 0$.

A special case of interest is the system consisting of a vortex with a non-rotating jet core ($\Omega^* = 0$); equation (28) becomes a quadratic equation for the phase velocity c^* and solving for c^*, one obtains

$$c^* = \frac{\Gamma^* n\dfrac{K_n(\alpha^*)}{K_n'(\alpha^*)}\dfrac{I_n'(\alpha^*)}{I_n(\alpha^*)}-\alpha^*\pm\sqrt{\Delta}}{\alpha^*\left(\dfrac{K_n(\alpha^*)}{K_n'(\alpha^*)}\dfrac{I_n'(\alpha^*)}{I_n(\alpha^*)}-1\right)}, \quad (29)$$

with
$$\Delta = \left(\Gamma^* n\frac{K_n(\alpha^*)}{K_n'(\alpha^*)}\frac{I_n'(\alpha^*)}{I_n(\alpha^*)}-\alpha^*\right)^2-\left(\alpha^*\frac{K_n(\alpha^*)}{K_n'(\alpha^*)}\frac{I_n'(\alpha^*)}{I_n(\alpha^*)}-\alpha^*\right)$$
$$\times\left(\frac{\Gamma^{*2}n^2}{\alpha^*}\frac{K_n(\alpha^*)}{K_n'(\alpha^*)}\frac{I_n'(\alpha^*)}{I_n(\alpha^*)}-\alpha^*+\Gamma^{*2}\frac{I_n'(\alpha^*)}{I_n(\alpha^*)}\right). \quad (30)$$

In this last case the condition for instability is simply that

$$\Delta < 0.$$

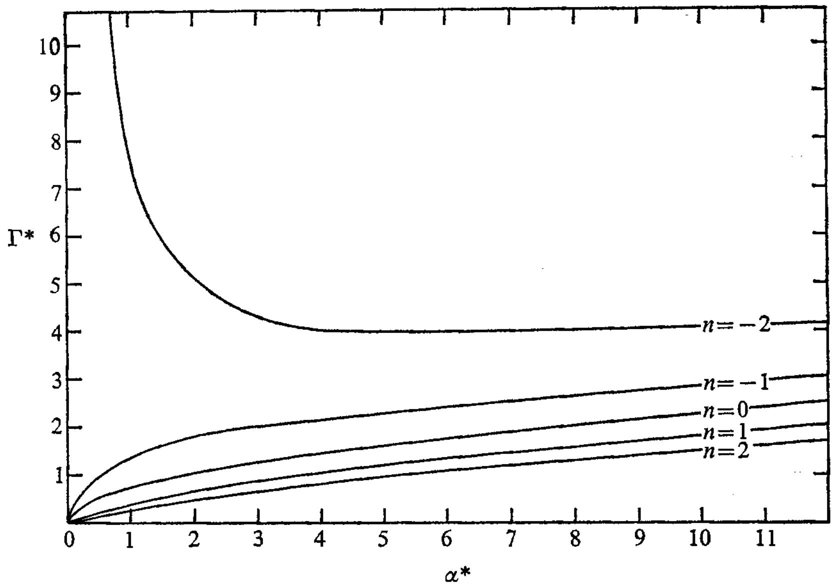


FIGURE 2. Neutral-stability curves for the non-rotating jet core ($\Omega^* = 0$).

3. Calculations and results†

The eigenvalue problem has been solved numerically. The growth rates have been computed for a non-rotating and rotating jet core with no discontinuity in the tangential velocity ($\Omega^* = \Gamma^*$) for different values of Γ^*.

For the rotating jet core, as the argument of the modified Bessel function I_n is complex and depends on the eigenvalue, an iterative procedure was used to determine the latter. The $n = 0$, ± 1 and ± 2 wave modes have been considered.

For the non-rotating jet, figure 1 shows the phase velocities and the growth rates for different values of Γ^*. As was indicated previously, the non-rotating jet becomes neutrally stable when Δ, as defined in (30), becomes equal to zero. This equation gives a relationship between Γ^* and α^* which can be used to obtain neutral-stability curves, which are shown in figure 2, for different values of n. From this figure, one can obtain the range of values of α^*, for given values of Γ^* and n, for which the flow is stable. Although it is not apparent from the neutral-stability curves, the flow always becomes unstable for all values of Γ^* and n in the limit $\alpha^* \to \infty$. This can be seen from (29), which shows that, in the limit $\alpha^* \to \infty$, $c^* = 1/2(1+i)$. This last result could have been anticipated as short waves are not influenced by the curvature and so behave as in Helmholtz instability.

Figure 3 shows the phase velocities and growth rates for the vortex with a rotating jet core. The flow is unstable for axisymmetric disturbances for all

† While this paper was in preparation, an article by Uberoi, Chow & Narain (1972) covering somewhat the same topic appeared. They derived approximate expressions for the eigenvalue c^* valid for very small and very large wavenumbers. In our paper all values of wavenumbers are explored along with rotating-core cases.

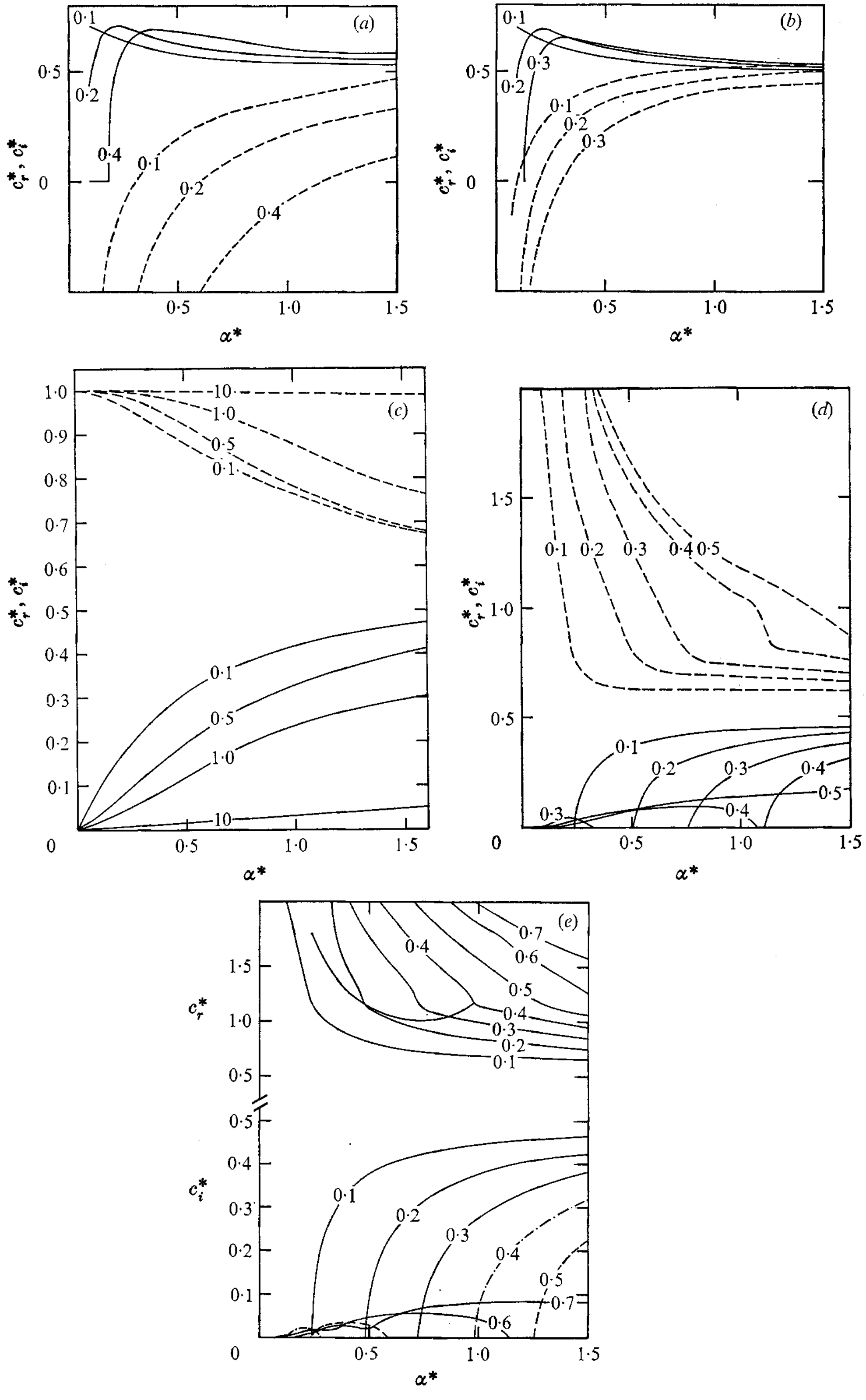


FIGURE 3. Phase velocity and growth rate for (*a*) $n = -2$, (*b*) $n = -1$, (*c*) $n = 0$, (*d*) $n = 1$ and (*e*) $n = 2$ for different values of Γ^* (numbers on curves) for the rotating core. (*a*)–(*d*) ---, c_r^*; ——, c_i^*.

values of Γ^* and α^*. In the non-axisymmetric case for small values of Γ^*, the flow is stable for small α^*. The value of α^* at which the flow becomes unstable increases as Γ^* increases. However, beyond a certain value of Γ^*, an additional range of wavenumbers at which the flow becomes unstable develops. This range also increases as Γ^* increases. This trend can be seen from figures 3 (*d*) and (*e*). In the range of stable wavenumbers, there are two or possibly more values of c^* that satisfy the eigenvalue equation; we have computed two real values of c^* for $\Gamma^* = 0{\cdot}4$ and $n = 2$. In both cases, modes with negative values of n are more unstable than those with positive n.

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